Solutions Midterm Exam — Complex Analysis

Aletta Jacobshal 03, Monday 15 December 2014, 09:00 - 11:00

Duration: 2 hours

Instructions

1. The test consists of 5 questions; answer all of them.
2. The number of points for each question is indicated at the beginning of the question. 10 points are "free" and the total number of points is divided by 10. The final grade will be between 1 and 10.

Question 1 (20 points)

Consider the function

$$f(z) = (x^2 + y^2) + 2ix,$$

where $z = x + iy$.

(a) Find the points (if any) in $\mathbb{C}$ where f is differentiable.

(b) Explain if f is analytic at the points you found in subquestion (a).

Solution

(a) Write

$$f(z) = u(x, y) + iv(x, y) = (x^2 + y^2) + i(2x),$$

and check the Cauchy-Riemann equations. We have

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \Leftrightarrow 2x = 0 \Leftrightarrow x = 0$$

and

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \Leftrightarrow 2y = -2 \Leftrightarrow y = -1.$$

Therefore the only point in $\mathbb{C}$ where f may be differentiable is $z = (0) + (-1)i = -i$. Furthermore, all partial derivatives exist and are continuous everywhere in $\mathbb{C}$ and in particular at $z = -i$. Therefore f is differentiable at $z = -i$.

(b) Although f is differentiable at $z = -i$, it is not analytic there since it is not differentiable in any open neighborhood of $z = -i$.

Question 2 (20 points)

It is given that the prinicipal value of the arc tangent of a complex number is

$$\operatorname{Tan}^{-1} z = \frac{i}{2} \operatorname{Log} \frac{1 - iz}{1 + iz}.$$

(a) Determine the principal value of the arc tangent of $2 + i$.

(b) For which complex numbers z is the function $\operatorname{Tan}^{-1} z$ discontinuous (or not defined) at z?

Solution

(a) We have

$$\frac{1 - iz}{1 + iz} = \frac{1 - i(2 + i)}{1 + i(2 + i)} = \frac{2 - 2i}{2i} = \frac{1 - i}{i} = \frac{1}{i} - 1 = -1 - i.$$

Then

$$\begin{aligned}
\operatorname{Tan}^{-1}(2 + i) &= \frac{i}{2} \operatorname{Log}(-1 - i) \\
&= \frac{i}{2} (\operatorname{Log} |-1 - i| + i \operatorname{Arg}(-1 - i)) \\
&= \frac{i}{2} \operatorname{Log} |-1 - i| - \frac{1}{2} \operatorname{Arg}(-1 - i) \\
&= \frac{i}{2} \operatorname{Log} \sqrt{2} - \frac{1}{2} \left(-\frac{3\pi}{4} \right) \\
&= \frac{i}{4} \operatorname{Log} 2 + \frac{3\pi}{8}.
\end{aligned}$$

(b) $\operatorname{Tan}^{-1} z$ is discontinuous or not defined whenever the argument w of Log has $\operatorname{Im} w = 0$ and $\operatorname{Re} w \leq 0$. Here, writing $z = x + iy$, we find

$$\begin{aligned}
w &= \frac{1 - iz}{1 + iz} = \frac{1 - i(x + iy)}{1 + i(x + iy)} = \frac{1 + y - ix}{1 - y + ix} = \frac{(1 + y - ix)(1 - y - ix)}{(1 - y)^2 + x^2} \\
&= \frac{1 - y^2 - x^2 - 2ix}{(1 - y)^2 + x^2}.
\end{aligned}$$

Therefore

$$\operatorname{Re} w = \frac{1 - y^2 - x^2}{(1 - y)^2 + x^2}, \qquad \operatorname{Im} w = \frac{-2x}{(1 - y)^2 + x^2}.$$

The condition $\operatorname{Im} w = 0$ gives $x = 0$. Then the condition $\operatorname{Re} w \leq 0$ gives

$$1 - y^2 \leq 0 \Leftrightarrow y^2 \geq 1 \Leftrightarrow y \geq 1 \text{ or } y \leq -1.$$

This means that $\operatorname{Tan}^{-1} z$ is discontinuous at z with $\operatorname{Re} z = 0$ and $\operatorname{Im} z \geq 1$ or $\operatorname{Im} z \leq -1$.

Question 3 (15 points)

Compute the integral

$$\int_\Gamma z^2\, dz,$$

where Γ is any contour in $\mathbb{C}$ that starts at 1 and ends at $-1+i$.

Solution

The function $z^3/3$ is an anti-derivative for z^2. Therefore

$$\int_\Gamma z^2\, dz = \frac{(-1+i)^3}{3} - \frac{1^3}{3} = \frac{1+2i}{3}.$$

Question 4 (20 points)

Compute the value of the integral

$$\int_\Gamma \frac{2e^z}{(z-1)^2(z-3)}\,dz$$

where Γ is the closed contour shown in Figure 1.

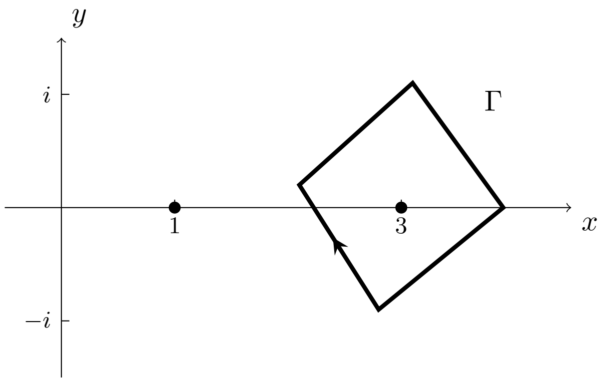

Figure 1: Contour Γ for Question 4.

Solution

The function

$$f(z) = \frac{2e^z}{(z-1)^2}$$

is analytic in the domain $D = \mathbb{C} \setminus \{1\}$ which includes the path Γ. Then

$$\int_\Gamma \frac{f(z)}{z-3}\,dz = -\int_{-\Gamma} \frac{f(z)}{z-3}\,dz = -2\pi i f(3) = -i\pi e^3.$$

Question 5 (15 points)

Consider a function $f(z)$ analytic in a domain D. Prove that if $|f(z)|^2$ is constant in D then the function $f(z)$ is constant in D.

Solution

Write $f(z) = u + iv$. Then $|f(z)|^2 = u^2 + v^2$. Since $|f(z)|^2$ is constant in D we have

$$\frac{\partial(u^2 + v^2)}{\partial x} = 0 \Rightarrow 2uu_x + 2vv_x = 0,$$

where $u_x = \partial u/\partial x$, $v_x = \partial v/\partial x$. Similarly,

$$\frac{\partial(u^2 + v^2)}{\partial y} = 0 \Rightarrow 2uu_y + 2vv_y = 0,$$

where $u_y = \partial u/\partial y$, $v_y = \partial v/\partial y$. The Cauchy-Riemann equations give

$$u_y = -v_x, \quad v_y = u_x.$$

Therefore we have the two relations

$$uu_x + vv_x = 0, \qquad vu_x - uv_x = 0.$$

Multiplying the first equation by u, the second by v and adding together we get

$$(u^2 + v^2)u_x = 0.$$

Then multiplying the first equation by v, the second by $-u$ and adding together we get

$$(u^2 + v^2)v_x = 0.$$

Recall that $u^2 + v^2$ is constant in D. Now we can distinguish two cases. If $u^2 + v^2 = 0$ then $u = v = 0$ so $f = u + iv = 0$ is also constant in D. The second case is $u^2 + v^2 = c > 0$. Then we must have $u_x = v_x = 0$. The Cauchy-Riemann equations give also $u_y = v_y = 0$. Therefore $u = c_1$ is constant in D and $v = c_2$ is constant in D, so

$$f(z) = u + iv = c_1 + ic_2,$$

is constant in D.